

THE CONVERSE OF BAER'S THEOREM

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ABSTRACT. The Baer theorem states that for a group G finiteness of $G/Z_i(G)$ implies finiteness of $\gamma_{i+1}(G)$. In this paper we show that if $G/Z(G)$ is finitely generated then the converse is true.

1. INTRODUCTION

A basic theorem of Schur (see [3, 10.1.4]) assert that if the center of a group G has finite index, then the derived subgroup of G is finite. This raises various questions: is there a generalization to higher terms of the upper and lower central series? Is there a converse? There has been attempts to modify the statement and get conclusions. Some authors studied the situation under some extra conditions on the group. For example B. H. Neumann [1] proved that $G/Z(G)$ is finite if $\gamma_2(G)$ is finite and G is finitely generated. This result is recently generalized by P. Niroomand [2] by proving that $G/Z(G)$ is finite if $\gamma_2(G)$ is finite and $G/Z(G)$ is finitely generated. For generalizing to higher terms of the upper and lower central series, R. Baer (see for example [3, 14.5.1]) has proved that, if $G/Z_i(G)$ is finite, then $\gamma_{i+1}(G)$ is finite. P. Hall (see for example [3, 14.5.3]) has proved a partial converse of Baer's theorem, that is, if $\gamma_{i+1}(G)$ is finite, then $G/Z_{2i}(G)$ is finite. In this paper we will prove that a converse of Baer's theorem when $G/Z(G)$ is finitely generated.

2. RESULTS

Theorem A. *Let G be a finitely generated group. $\gamma_{i+1}(G)$ is finite if and only if $G/Z_i(G)$ is finite.*

Proof. Let $a \in G$, since $\gamma_{i+1}(G) = [\gamma_i(G), G]$ is finite, the set of conjugates $\{a^b : b \in \gamma_i(G)\}$ is finite, so $C_{\gamma_i(G)}(a)$ has finite index in $\gamma_i(G)$. Since G is finitely generated, $\frac{\gamma_i(G)}{(\gamma_i(G) \cap Z(G))}$ is finite. Hence $\frac{(\gamma_i(G)Z(G))}{Z(G)} = \gamma_i(G/Z(G))$ is finite. So by induction $\frac{(G/Z(G))}{Z_{i-1}(G/Z(G))}$ is finite, and then $G/Z_i(G)$ is finite. Now 14.5.1 of [3] completes the proof. \square

Lemma 2.1. *Let G be a group and $G/Z(G) = \langle x_1Z(G), \dots, x_nZ(G) \rangle$. Then $Z(G) = \bigcap_{i=1}^n C_G(x_i)$.*

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Proof. It is clear that, $Z(G)$ is a subset of $C_G(x)$, for any $x \in G$. Now let a be an element of G such that, $[a, x_i] = 1$ for $i = 1, \dots, n$. For any $b \in G$, $b = y_1 y_2 \cdots y_t z$ where, $y_i \in \{x_1, \dots, x_n\}$ and $z \in Z(G)$. Now we have $[a, b] = [a, y_1 y_2 \cdots y_t z] = 1$, since $[a, y_i] = 1 = [a, z]$. Therefore a is an element of $Z(G)$. \square

Theorem B. *Let G be a group and $G/Z(G)$ finitely generated. Then $\gamma_{i+1}(G)$ is finite if and only if $G/Z_i(G)$ is finite.*

Proof. If $i = 1$, then the main theorem of [2] implies the result. Let $i > 1$ and $G/Z(G) = \langle x_1 Z(G), \dots, x_n Z(G) \rangle$. As the proof of theorem A, $C_{\gamma_i(G)}(a)$ has finite index in $\gamma_i(G)$ for any $a \in G$. Since $Z(G) = \bigcap_{i=1}^n C_G(x_i)$, $\frac{\gamma_i(G)}{(\gamma_i(G) \cap Z(G))}$ is finite. Hence $\frac{(\gamma_i(G)Z(G))}{Z(G)} = \gamma_i(G/Z(G))$ is finite. Now $\frac{G/Z(G)}{Z(G/Z(G))} = \frac{G}{Z_2(G)}$ is finitely generated, so by induction $\frac{(G/Z(G))}{Z_{i-1}(G/Z(G))}$ is finite, and then $G/Z_i(G)$ is finite. Now 14.5.1 of [3] completes the proof. \square

The following example shows the finiteness conditions on the Theorem B is necessary.

Example 1. Let G be a group with generators $x_j, y_j, j > 1$ and z , subject to the relations $x_j^p = y_j^p = z^{p^i} = 1$, $[x_l, x_j] = [y_l, y_j] = 1$, for $k \neq j$, $[x_k, y_j] = 1$, $[x_j, y_j] = z$ and $[z, t_1, \dots, t_r] = z^{p^r}$ where $t_s \in \{x_j, y_j\}$ for $s = 1, \dots, r$ and $1 \leq r \leq i-1$. Then $Z_i(G) = \langle z \rangle$ and $\gamma_{i+1}(G) = \langle z^{p^{i-1}} \rangle$, but $G/Z_i(G)$ is infinite.

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